Chapter 5: Strength of Materials

Introduction
Statics (covered in Chapters 2 - 4) is essentially force analysis - the determination of the internal forces developed in members of a structural framework by externally applied loads.

• Statics in itself is not the design of any member, but it is a first step leading to structural design.

• The objective of a study in strength (mechanics) of materials is to develop the relationship between loads applied to a non-rigid body and the resulting internal forces and deformations.

• The internal forces (together with known allowable stresses) are used to determine the size of a structural element required to resist safely the externally applied loads.

• The basis of structural design is calculating forces, selecting materials based on material properties, and determining appropriate structural sizes.

5.1 Stress and Strain
Chapters 5 - 9 establish the methods for solving three general types of problems.
1. Design
   • Given a certain function to perform, of what materials should the structure be constructed?
   • What should be the sizes and proportions of the various elements?

2. Analysis
   • Given the completed design, is it adequate?
   • Does it perform the function economically and without excessive deformation?
   • What is the margin of safety allowed in each member?

3. Rating
   • Given a completed structure, what is the actual load-carrying capacity?
   • Is the structure or its members adequate for the proposed new use?
      - The structure may have been designed for some purpose other than the one for which it is now used.
Structural Load Classification
Loads applied to structural elements may be of various types and sources.

- Loads classified with respect to time.
  1. Static load
     - A gradually applied load; equilibrium is reached in a relatively short time.
     - Live or occupancy loads are considered statically applied.
  2. Sustained load
     - A load that is constant over a long period of time.
     - The structure weight (dead load) or material and/or goods stored in a warehouse are considered sustained loads.
     - This type of load is treated in the same manner as a static load.
  3. Impact load
     - A load that is rapidly applied (an energy load).
     - Vibration normally results from an impact load, and equilibrium is not established until the vibration is eliminated, usually by natural damping forces.

- Loads classified with respect to the area over which the load is applied.
  1. Concentrated load
     - A load or force that is applied at a point.
     - A concentrated load is an idealization.
     - Realistically, any load that is applied to a relatively small area compared with the size of the loaded member is considered a concentrated load.
  2. Distributed load
     - A load distributed along a length or over an area.
     - The load distribution may be uniform or non-uniform.
• Loads classified with respect to the location and method of application.

1. Centric (concentric) load
   • A load that passes through the centroid (i.e. geometric center) of the cross section.
     - Force P, shown at the right, has a line of action that passes through the centroid of the column and the footing; therefore, load P is axial.
   • The load is called an axial load if the force passes through the centroids of all sections.

2. Bending or flexural load
   • A load that is applied transversely to the longitudinal axis of the member.
   • The load may include applied couples.
   • A member subjected to bending loads deflects along its length.

   The figure above illustrates a beam subjected to flexural load consisting of a concentrated load, a uniformly distributed load, and a couple.

3. Torsional load
   • A load that subjects a member to couples or moments that twist the member spirally.

4. Combined loading
   • A combination of two or more of the previously defined types of loads.

Concept of Stress

*Stress* is a term used to describe the intensity of a force.

• Stress is the quantity of force that acts on a unit of area.

• Force, in structural design, has little significance until something is known about the resisting material, cross-sectional properties, and the size of the element resisting the force.
Axial Stress
The average value of the axial stress may be represented mathematically (i.e. mathematical model) as follows.
\[ f = \sigma = \frac{P}{A} \]
where
\[ \sigma = \text{unit stress (psi, ksi, ksf, N/m}^2, \text{kN/m}^2, \text{or N/mm}^2) \]
\[ P = \text{applied force or load (axial) (lbs, kips, N, or kN)} \]
\[ A = \text{resisting cross-sectional area perpendicular to the load direction (in}^2, \text{ft}^2, \text{m}^2 \text{or mm}^2) \]
The unit stress on any cross section of an axially loaded, two-force member is considered to be uniformly distributed unless otherwise noted.

Normal Stress
Axial tension or compressive forces produce tensile or compressive stress, respectively.
- This type of stress is classified as a normal stress because the stressed surface is normal (perpendicular) to the load direction.
- The average value of the normal stress may be represented mathematically as follows.
  - Normal compressive stress: \[ f_c = \frac{P}{A} \]
  - Normal tensile stress: \[ f_t = \frac{P}{A} \]
where
\[ P = \text{applied load} \]
\[ A = \text{resisting surface normal (perpendicular) to P} \]

Shear Stress
Shear stress is caused by a force applied to an area that is parallel to the direction of the applied load.
- The average shear stress may be represented mathematically as follows.
\[ f_v = \tau = \frac{P}{A} \]
where
\[ P = \text{applied load (lb, k, N, or kN)} \]
\[ A = \text{cross-sectional area parallel to load direction (in}^2, \text{ft}^2, \text{m}^2 \text{or mm}^2) \]
\[ f_v \text{ or } \tau = \text{average unit shear stress (psi, ksi, ksf, N/m}^2, \text{kN/m}^2, \text{or N/mm}^2) \]
Shear stresses act on the cross-sectional area of the bolt(s) connecting two steel plates.

- A bolted connection in single shear.
  \[ f_v = \frac{P}{A} \]
  ![Single shear plane](image)

- A bolted connection in double shear (two shear planes).
  \[ f_v = \frac{P}{2A} \]
  ![Two shear planes](image)

where
\[ f_v = \text{average shear stress through the cross-section of the bolt} \]
\[ A = \text{cross-sectional area of one bolt} \]

**Bearing Stress**

*Bearing stress*, a type of normal stress, represents the intensity of force between a body in contact with another body.

- Common examples of *bearing stress* include the following.
  - Contact between a bolt and bar.
  - Contact between a beam and column.
  - Contact between a column and a footing.
  - Contact between a footing and the ground.

The stressed surface is perpendicular to the direction of the applied load (like the normal stress).

The average bearing stress may be represented mathematically as follows.
\[ f_p = \frac{P}{A} \]

where
\[ f_p = \text{unit bearing stress (psi, ksi, ksf, N/m}^2\text{, kN/m}^2\text{, or N/mm}^2\text{)} \]
\[ P = \text{applied load (lb, k, N, or kN)} \]
\[ A = \text{bearing contact area (in}^2\text{, ft}^2\text{, m}^2\text{ or mm}^2\text{)} \]
In the preceding three stress classifications, the basic equation of stress may be written in three different ways, depending on the conditions being evaluated.

1. \( f = \frac{P}{A} \)
   - The basic equation; used for analysis purposes in which the load, member size, and material are known.

2. \( P = f \times A \)
   - Used in evaluating or checking the capacity of a member when the material and member size are known.

3. \( A = \frac{P}{f} \)
   - Design version of the stress equation.
   - The member size can be determined if the load and material's allowable stress are known.

**Torsional stress**

Structural members are subjected *torsional stress* by the twisting along their longitudinal axis by a moment couple or eccentric load.

- Most building members subjected to torsional effects are also experiencing either bending, shear, tensile, and/or compressive stresses.
- It is uncommon to design specifically for torsion.
- However, designs involving machinery and motors with shafts are extremely sensitive to the stresses resulting from torsion.
Example Problems - Stress

Problem 5.1 (p. 262)

Given: System shown.
   Member AB is $\frac{1}{2}''$ thick x 2'' wide.

Find: Tensile stress in member AB.

Solution

Find force in member AB.

FBD: Member BCD

\[ \Sigma M_C = 0 = 500 (3) - (1/\sqrt{2}) \text{ AB} (2) \]

\[ 1.414 \text{ AB} = +1500 \]

\[ \text{AB} = 1060.7 \text{ lb (tension)} \]

Compute the stress in member AB.

\[ f_t = \frac{P}{A} = \frac{1060.7}{0.5 (2)} = 1060.7 \text{ psi} \]

Note: The allowable stress in an A36 steel rod: \( F_t = 22,000 \text{ psi} \).
Problem 5.2 (p. 262)

Given: Hotel marquee shown (10’ x 20’).

- \( F_t = 22,000 \text{ psi (allowable stress)} \)
- \( DL + \text{Snow load} = 100 \text{ psf} \)

Find: Design the two rods \( AB \) (A36 steel).

Solution

Find the force in each steel rod.

\[
\sum M_C = 0 = 2[- AB \sin 30^\circ (10)] + W (5)
\]

\[
0 = - 2AB \sin 30^\circ (10) + (100 \text{ psf}) (10) (20) (5)
\]

\[ AB (10.0) = 100,000 \]

\[ AB = 10,000 \text{ lb} \]

Determine the diameter of each steel rod.

\[ A_{reqd} = \frac{P}{F_t} = \frac{10,000}{22,000} = 0.454 \]

\[ A_{reqd} = 0.454 \text{ in}^2 \]

For a circle: \[ A = \pi \frac{d^2}{4} \] and \[ d = \sqrt{\frac{4A}{\pi}} \]

Required rod diameter: \[ d = \sqrt{4(0.454)/\pi} \approx 0.760'' \] (Use 13/16”)

The actual area of a 13/16” rod is 0.5185 \( \text{in}^2 > 0.454 \text{ in}^2 \) OK

Check the actual stress in each rod.

\[ f_t = \frac{P}{A} = \frac{10,000}{0.5185} = 19,286 \text{ psi} < F_t = 22,000 \text{ psi} \] OK
Deformation and Strain
Most materials of construction deform under the action of loads (i.e., not a true rigid body).
- When the size or shape of a body is altered, the change in any direction is termed deformation and given by the symbol $\delta$ (delta).
- Strain, $\varepsilon$ (epsilon) or $\gamma$ (gamma), is defined as the deformation per unit length.
- The deformation or strain may be the result of a change of temperature or stress.

Consider a piece of rubber that is being stretched.
- The rubber tends to elongate in the direction of the applied load with a resultant deformation $\delta$. Correspondingly, a contraction of the width occurs.
  \[ L = \text{original length} \]
  \[ W = \text{original width} \]
  \[ W' = \text{new width} \]
  \[ \delta_L = \text{longitudinal change in length (deformation)} \]
  \[ W - W' = \delta_t = \text{transverse change in length} \]
- Strain resulting from a change in stress is defined mathematically as follows.
  \[ \varepsilon = \frac{\delta}{L} \]
  where
  \[ \varepsilon = \text{unit strain (inches/inch)} \]
  \[ \delta = \text{total deformation (inches)} \]
  \[ L = \text{original length (inches)} \]
Members subjected to a shear stress undergo a deformation that results in a change in shape.

- Shear stress causes an angular deformation of the body rather than an elongation or shortening.
- The square shown at the right becomes a parallelogram when acted upon by shear stresses.

Shearing strain, represented by $\gamma$, is defined mathematically as follows.

$$\gamma = \frac{\delta_s}{L} = \tan \phi \approx \phi$$

When the angle $\phi$ is small, $\tan \phi \approx \phi$, where $\phi$ is the angle expressed in radians.
Example Problems - Deformation and Strain

Problem 5.7 (p. 266)

Given: Test specimen shown.
\[ L = 2'' \]
\[ \delta = 0.0024'' \]

Find: Strain

Solution
\[ \varepsilon = \frac{\delta}{L} = \frac{0.0024}{2} = 0.0012 \text{ in/in} \]

Problem 5.9

Given: Concrete test cylinder shown.
\[ L = 8'' \]
\[ \varepsilon = 0.003 \text{ in/in} \]

Find: Shortening that develops as a result of the load.

Solution
\[ \varepsilon = \frac{\delta}{L} \]
\[ \delta = \varepsilon L = 0.003 (8) = 0.024'' \]

Notes:
1. The answer given in the textbook is \( \delta = 0.0264'' \). The textbook's answer is based on the strain \( \varepsilon = 0.0033 \text{ in/in} \).
2. The 4'' diameter is not involved in the analysis.
5.2 Elasticity, Strength, and Deformation

**Relationship Between Stress and Strain**
A variety of materials is used in architectural structures.
- Materials include stone, brick, concrete, steel, timber, aluminum, and plastics.
- These materials have properties that allow them to be used in a structure.
- A material is selected for use based on its ability to withstand forces without excessive deformations or actual failures.

A major consideration that must be considered in any structural design is deflection (*deformation*).
- Deformation in structures cannot increase indefinitely without failure of the structural element.
- Deflection should disappear after the applied load is removed.
- *Elasticity* is a material property in which deformations disappear once the load is removed.

Deformations will disappear after the load is removed as long as a certain stress limit (i.e. *elastic limit*) is not exceeded.
- If stresses remain below the *elastic limit*, then no permanent deformations result from the application and removal of a load.
- If the *elastic limit* is exceeded, then permanent deformation results.
- When permanent deformations remain, the behavior of the material is said to be *plastic* or *inelastic*.
Brittle materials
• When the elastic limit is exceeded in brittle materials, the molecular bonds within the material are unable to reform.
• Cracks form or there is separation of the material.
• Cast iron, high-carbon steel, and ceramics are considered brittle materials.
• Brittle materials give no warning of impending failure.

Ductile materials
• When the elastic limit is exceeded in ductile materials, the molecular bonds reform.
• Permanent deformations result, but the material remains intact without a significant loss of strength.
• Low-carbon steel, aluminum, copper, and gold are considered ductile materials.
• Ductile materials give warning of impending failure.

Plasticity
• Plasticity describes the deformation of a material undergoing non-reversible changes of shape in response to applied forces.
• A solid piece of metal or plastic being bent or pounded into a new shape displays plasticity as permanent changes occur within the material itself.
• In engineering, the transition from elastic behavior to plastic behavior is called yielding.

One of the most important discoveries in the science of mechanics of materials pertains to the elastic behavior of materials.
• The discovery was made in 1678 by Robert Hooke, an English scientist.
• Hooke’s discovery mathematically relates stress to strain.
• Hooke’s law states that stress and strain are proportional in elastic materials.
Universal testing machines are used to apply precise loads at precise rates to standardized tensile and compressive test specimens.

- The tensile test is the most common test.
- Devices are used for measuring and recording strain or deformation.
- The data obtained from the tests are used to plot stress-strain diagrams (or load-deformation curves).

Figure 5.20 (page 269 of the textbook) shows a stress-strain diagram for various materials.

- Several characteristic patterns are revealed.
  - Ductile rolled steel (ordinary, low-carbon structural steel) follow a straight-line variation and then deform considerably after the elastic limit is exceeded.
  - Materials such as cast iron, brass, concrete, and wood follow a curved-line through most of their length and there is no distinct elastic limit.

Stress-strain diagram

- The stress-strain diagram plots strain along the abscissa (horizontal) and stress along the ordinate (vertical).
- The stress is defined as the load (pounds or kips) divided by the original cross-sectional area of the test specimen.

As the test proceeds, larger loads are applied at a specified rate.

- The strain determined during the test is based on a test specimen with an original gauge length of 2" and a diameter of \( \frac{1}{2} \)".
- The actual cross-sectional area of the specimen decreases.
- At high stresses, this reduction in the cross-sectional area of the test specimen becomes significant.
- The calculated stress (called the indicated stress) that is determined during the test is based on the original cross-sectional area of the test specimen and is not the true stress.

Figure 5.22 (page 270 of the textbook) shows the stress-strain curve for mild steel (A36).

- The significant points on the stress-strain curve are defined as follows.
  1. Proportional limit
- The *proportional limit* is the limit of the linear variation of stress and strain.
- Hooke’s law of stress-strain proportionality is no longer valid when stresses exceed this limit.

2. Elastic limit
- The *elastic limit* is close to the proportional limit.
- The elastic limit is the maximum stress that can be developed in a material without causing a permanent set (deformation).
- A specimen stressed to a point below its elastic limit will return to its original dimensions when the load is released.
- If the stress should exceed the *elastic limit*, the specimen will deform plastically and will not return to its original dimensions when the load is released. The material is said to have a permanent set.

3. Yield point
- The *yield point* is the stress at which strain (or deformation) continues to increase without an increase in the applied load.
- After the initial yielding (upper yield point) is reached, the force resisting deformation actually decreases due to the yielding of the material.
- The value of stress after the initial yield point (known as the *lower yield point*) is usually used as the basis for determining the allowable stress that is used for design purposes.

Some materials (such as cast iron) do not exhibit a well-defined yield point.
- The yield strength is defined as the stress at which the material exhibits a specified limiting permanent set.
- The specified set (or offset) most commonly used is 0.2%, which corresponds to a strain of 0.002 in/in.

4. Ultimate strength
- The *ultimate strength* (stress) of a material is the maximum load (i.e. the load reached when the specimen breaks) divided by the original cross-sectional area.
- The *ultimate strength* (a.k.a. the tensile strength) of a material is sometimes used as a basis for establishing the allowable design stresses.
5. Rupture strength
   - *Rupture strength* is the breaking strength, or fracture strength of the material.
   - In a ductile material, rupture does not usually occur at the ultimate stress.
     - After the ultimate stress has been reached, the material will generally neck down, and its rapidly increasing elongation will be accompanied by a decrease in load.
   - The *rupture strength*, is determined by dividing the load at rupture by the original cross-sectional area of the specimen.
   - Rupture strength has little or no value in design.

6. Reduction of area.
   - As the load on the tested material is increased, the original cross-sectional area decreases to a minimum at the instant of fracture.
     - The failed specimen exhibits a local decrease in diameter known as *necking down* in the region where the failure occurs.
     - Ductile materials exhibit a high reduction in area; brittle materials exhibit almost no reduction in area.
   - The true value of stress, following the ultimate stress, is determined by dividing the loads by the actual (decreasing) cross-sectional areas.

5.3 Other Material Properties

*Compression Tests*
- The compression test is used primarily to test brittle materials such as cast iron and concrete.
- The results of the compression test generally define an elastic range, a proportional limit, and a yield strength.
- In the compression test, the cross-sectional area of the specimen increases as the load increases.

*Poisson's Ratio*
When a material is loaded in one direction, it will undergo strains perpendicular to the direction of the load as well as parallel to it.
- The ratio of the lateral (perpendicular) strain to the longitudinal (axial) strain is called *Poisson's ratio*. 
• Poisson’s ratio varies from 0.2 to 0.4 for most metals.
  - Most steels have values in the range of 0.283 to 0.292.
• The symbol μ (mu) is used for Poisson’s ratio, which is given by the equation.
  \[ \mu = \frac{\varepsilon_{\text{lateral}}}{\varepsilon_{\text{longitudinal}}} \]

**Allowable Stress - Factor of Safety**

An allowable stress is defined as the maximum stress that is permitted in a design calculation.

• The allowable stress for A36 steel \((F_y = 36 \text{ ksi})\) in tension is expressed as follows.
  \[ F_t = 22.0 \text{ ksi} \]

The factor of safety is defined as the ratio of a failure-producing load to the estimated actual load.

  \[ \text{F.S.} = \frac{\text{Failure load}}{\text{Actual load}} \]

**Modulus of Elasticity (Young’s Modulus)**

In 1678 Sir Robert Hooke observed that stress is proportional to strain.

In 1807 Thomas Young suggested that this modulus (ratio) could be used as a means of evaluating the stiffness of materials.

• The modulus of elasticity (called Young’s modulus) is found by dividing stress by strain.
• This ratio is the slope of the straight-line portion of the stress-strain diagram and is expressed mathematically as follows.
  \[ E = \frac{f}{\varepsilon} \]

where
  \[ E = \text{modulus of elasticity (ksi, psi, N/mm}^2\) \]
  \[ f = \text{stress (ksi, psi, N/mm}^2\) \]
  \[ \varepsilon = \text{strain (in/in, mm/mm)} \]

This ratio of stress to strain is constant for all steels and many other structural materials (e.g. \(E = 29,000 \text{ ksi for steel}\)).

• A high modulus of elasticity is desirable for structural materials.
  - \(E\) is often referred to as a stiffness factor.
Materials exhibiting high E values are more resistant to deformation.
- In the case of beams, the higher the E value, the less the deflection under load.

The equation for Young’s modulus may be written in the following form whenever the stress and deformation are caused by axial loads.

From previous work: \( f = \frac{P}{A} \) and \( \varepsilon = \frac{\delta}{L} \)

\[
E = \frac{f}{\varepsilon} = \frac{P}{A} = \frac{PL}{\delta/L} = \frac{\delta}{\delta A}
\]

From the equation for Young’s modulus, the equation for deformation due to axial loads is formulated.

\[
\delta = \frac{PL}{AE} \quad \text{(elastic equation)}
\]

where
- \( \delta \) = deformation (in, mm)
- \( P \) = applied axial load (lb, kips, N, or kN)
- \( L \) = length of member (in, mm)
- \( A \) = cross-sectional area of member (in\(^2\), mm\(^2\))
- \( E \) = modulus of elasticity of material (psi, ksi, N/mm\(^2\))
Example Problems - Material Properties

Problem 5.11 (p. 285)

Given: Garage structure shown.

Loadings: Roof = 100 psf
          Snow load = 30 psf
          Brick = 120 pcf

Allowable stress: $\sigma_{\text{brick}} = 125$ psi

Find: Compressive stress at the base of the wall.

Solution

Determine the load for a 1-foot strip of wall.

- Tributary width of roof area to each wall = $\frac{1}{2} \times 20' = 10'$

  $$P = \text{Roof load} + \text{Snow load} + \text{Weight of wall}
  = 100 \text{ psf} \times (10' \times 1') + 30 \text{ psf} \times (10' \times 1') + 120 \text{ pcf} \times (12' \times 1' \times 4''/12)
  = 1000 \text{ lb} + 300 \text{ lb} + 480 \text{ lb} = 1780 \text{ lb}$$

Determine the compressive stress at the base of the wall.

$$\sigma_{\text{bearing}} = \frac{P}{A} = \frac{1780}{(12'' \times 4'')} = 37.1 \text{ psi} < 125 \text{ psi} \quad \text{OK}$$
Problem 5.13 (p. 285)

Given: Steel rod 1.5” dia x 25’ long
Load = 29 kips
\(E_{\text{steel}} = 29 \times 10^3\) ksi

Find: a) Total elongation of rod
b) Required diameter to limit total elongation to 0.1”

Solution

Determine the total elongation using the elastic equation.

\[
\delta = \frac{PL}{AE} = \frac{29 (25') (12'')}{[\pi (1.5)^2/4] (29 \times 10^3)} = \frac{8,700}{(1.767) (29 \times 10^3)}
\]

\[
\delta = \frac{8,700}{51,243} = 0.1698”
\]

Determine the required diameter to limit the elongation to 0.1”.

\[
\delta = \frac{PL}{AE}
\]

\[
A = \pi \frac{d^2}{4} \quad \text{for a circular section}
\]

\[
0.1 = \frac{29 (25') (12'')}{[\pi (d)^2/4] (29 \times 10^3)} = \frac{8,700}{(0.7854) (d)^2 (29 \times 10^3)}
\]

\[
d^2 = \frac{8,700}{(0.7854) (0.1) (29 \times 10^3)} = 3.8197
\]

\[
d = 1.954”
\]
Stress Concentrations

Stress is generally considered as uniformly distributed on any cross-section of the member.

- We assume that a load is applied centrically (i.e. through the axis of the member).
- Generally, this is a practical assumption to make for a static load condition.

However, stress cannot be considered as uniformly distributed in the following cases.

- The geometry of the member changes to include discontinuities or changing cross sections.
- Load concentrations (e.g. corners, notches, openings, and other discontinuities) will cause stress concentrations.

Load concentrations do not necessarily produce structural failures, even if the maximum stress exceeds the allowable working stress.

- In structural steel, extreme stress conditions may be relieved because steel is ductile and has a tendency to yield (give).
  - The stress is redistributed over more of the cross-sectional area.
  - The redistribution of stress enables the greater part of the structural member to be within permissible stress range.
- In concrete, a stress concentration is a serious matter.
  - Excessive tensile stresses, even though localized, cause cracks to appear in the concrete.
  - Over time, the cracks become more pronounced because of the high stress concentration at the end of the cracks.
  - Cracking in reinforced concrete can be minimized by placing the reinforcing steel across potential crack lines.
- In timber, stress concentrations cause cracks to appear along the grain.

In statically loaded ductile materials, stress concentrations are usually not critical.

- The material will yield inelastically in the high-stress areas.
- Redistribution of stress results.
- Equilibrium is established and no harm is done.
In dynamic or impact loading of ductile materials, or static loading of brittle materials stress concentrations become critical.

- Stress concentrations become very critical and cannot be ignored.
- Stress redistribution does not result to such an extent that equilibrium is maintained.

**Torsional Stress**

Charles-Augustin de Coulomb, a French engineer of the 18th century:

- He was the first to explain torsion in a solid or hollow circular shaft.
- He developed a relationship between the applied *torque* (T) and the resulting deformation (angle of twist) of circular rods.

When a torque (twisting moment) is applied to a circular shaft, a distortion of the rod occurs as shown at the right.

- Shearing stresses $f_v$ exist.
- In an elastic material (e.g. steel), the stresses increase in magnitude proportionately to the distance from the center of the circular cross section.

Coulomb derived the following relationship between the applied torque and stress using equilibrium concepts.

- For a solid circular shaft or a hollow circular shaft, the shear stress due to torsion is determined by the following equation.

\[ f_v = \frac{T (d/2)}{I_p} \tag{Note: This equation is not in the textbook.} \]

where

- $T$ = externally applied torsional moment (torque)
- $I_p$ = polar moment of inertia
  - $= (\pi/32) (d_o^4 - d_i^4)$ for hollow circular shafts
  - $= \pi d_i^4/32$ for solid circular shafts
- $d$ = the distance from the center of the shaft to the stress location
  - $= d_o$ for the maximum stress
- $d_o$ = the outside diameter of a solid or hollow circular shaft
- $d_i$ = the inside diameter of a hollow circular shaft
For a solid circular shaft only (where \( I_p = \pi d^4 / 32 \)), the shear stress due to torsion is determined by the following equation.

\[
f_v = \frac{T (d/2)}{I_p} = \frac{T (d/2)}{(\pi d^4 / 32)} = 16T/(\pi d^3) = 16T/\pi(2r)^3 = 16T/8\pi r^3
\]

\[
f_v = 2T/\pi r^3 \quad \text{or} \quad T = \pi r^3 f_v / 2
\]

where

- \( T \) = externally applied torsional moment (torque)
- \( \pi r^2 \) = cross sectional area of the rod (shaft)
- \( r \) = radius of the rod
- \( f_v \) = internal shear stress on the transverse plane of the rod

Hollow, circular cross sections (pipes) offer the greatest torque resistance per unit volume of material, since material located near the center in a solid circular rod is at low stress level and thus less effective.

Noncircular cross-sectioned members such as rectangular or I-shaped beams develop a completely different distribution of shear stress when subjected to torsion.
Example Problems - Torsional Stress

**Solid Circular Rod**

*Given:* Solid circular shaft shown.
- Applied torque = 500 lb-in

*Find:* Maximum internal shear stress.

**Solution**

Applicable equation: \( f_v = \frac{2T}{\pi r^3} \)

\[
f_v = \frac{2T}{\pi r^3} = \frac{2 \times 500 \text{ lb-in}}{\pi \times (0.50''/2)^3} = 20,372 \text{ lb/in}^2
\]

\( f_v = 20.4 \text{ ksi} \)

**Alternate solution**

Applicable equation: \( f_v = \frac{T(d/2)}{I_p} \)

\[
I_p = \left(\frac{\pi}{32}\right) \left(d^4 - d_i^4\right) = \left(\frac{\pi}{32}\right) \left[(1.0)^4 - (0.75)^4\right] = 0.006136 \text{ in}^4
\]

\[
f_v = \frac{T(d/2)}{I_p} = \frac{500 \text{ lb-in} \times (0.50''/2)}{0.006136 \text{ in}^4} = 20,372 \text{ lb/in}^2
\]

\( f_v = 20.4 \text{ ksi} \)

**Hollow Circular Rod**

*Given:* Hollow circular shaft shown.
- Applied torque = 750 lb-in

*Find:* Maximum and minimum internal shear stress.

**Solution**

Applicable equation: \( f_v = \frac{T(d/2)}{I_p} \)

\[
I_p = \left(\frac{\pi}{32}\right) \left(d_2^4 - d_i^4\right) = \left(\frac{\pi}{32}\right) \left[(1.0)^4 - (0.75)^4\right] = 0.06711 \text{ in}^4
\]

\[
(f_v)_{\text{max}} = \frac{T(d/2)}{I_p} = \frac{750 \text{ lb-in}(1.0''/2)}{0.06711 \text{ in}^4} = 5,588 \text{ lb/in}^2
\]

\( (f_v)_{\text{max}} = 5.59 \text{ ksi} \)

\[
(f_v)_{\text{min}} = \frac{T(d/2)}{I_p} = \frac{750 \text{ lb-in}(0.75''/2)}{0.06711 \text{ in}^4} = 4,191 \text{ lb/in}^2
\]

\( (f_v)_{\text{min}} = 4.19 \text{ ksi} \)
5.4 Thermal Effects
Most structural materials increase in volume when subjected to heat and contract when cooled.
• Whenever a design (i.e. a support or connection) prevents the change in length of a member subjected to temperature variation, internal stresses develop.
• These thermal stresses may be sufficiently high to exceed the elastic limit of the material and may cause serious damage.
• Free, unrestrained members experience no stress changes with temperature changes, but dimensional changes result.

The dimensional change due to temperature changes is usually described in terms of change in a linear dimension.
• The change in length of a structural member, $\Delta L$, is proportional to both temperature change ($\Delta T$) and the original length of the member $L_o$.
• Thermal sensitivity (i.e. the coefficient of linear expansion, $a$) has been determined for all engineering materials.
• Careful measurements show that the thermal sensitivity, $a$, is equal to the ratio of strain $\varepsilon$ to temperature change $\Delta T$.
• Thermal sensitivity is a constant value for a given material.

$$a = \frac{\text{Strain}}{\text{Temperature change}} = \frac{\varepsilon}{\Delta T} = \frac{(\delta/L)}{\Delta T}$$

Solve this equation for the deformation.
$$\delta = a L \Delta T$$
where
a = coefficient of thermal expansion
L = original length of member (inches)
$\Delta T$ = change in temperature ($^\circ$F)
$\delta$ = total change in length (inches)

Stresses are developed by restraining the free expansion and contraction of members subjected to temperature variations.
• To calculate these thermal stresses, first, determine the free expansion or contraction of the member.

$$\delta = a L \Delta T$$
Then, determine the force and unit stress developed in forcing the member to attain its original length.

\[ \delta = \frac{PL}{AE} = f \frac{L}{E} \]

The problem from this point on is the same as those solved in the earlier portions of this chapter dealing with axial stresses, strains, and deformations.

- The stress developed by restoring a bar to its original length \( L \) may be determined by the following equation.

\[ f = \varepsilon E = \left( \frac{\delta}{L} \right) E = \left( \alpha \frac{L}{\Delta T/L} \right) E \]

\[ f = \alpha \Delta T E \]

- Alternatively, the equation may be developed by equating the deformation due to temperature change with the deformation due to the axial load.

\[ \delta = \alpha L \Delta T \quad \text{deformation due to temperature change} \]

\[ \delta = f \frac{L}{E} \quad \text{deformation due to axial load} \]

Then, \( \alpha L \Delta T = f \frac{L}{E} \)

and \( f = \alpha L \Delta T \frac{E}{L} = \alpha \Delta T E \)
Example Problems - Thermal Effects

Problem 5.16 (p. 293)

Given: Long concrete bearing wall with vertical expansion joints placed every 40'

$\alpha = 6 \times 10^{-6}/^\circ F$

Find: Width of gap in joint to be wide open at 20° F and just barely closed at 80° F.

Solution

$\delta = \alpha L \Delta T = (6 \times 10^{-6}) (40') (12"/) (80° - 20°) = 0.1728"$  (Use 3/16")

Use a gap width of 3/16"
Problem 5.17 (p. 293)

Given: 12’ aluminum curtain wall supported by concrete columns, installed at 65°

- Aluminum wall panel heats up to 120°
- Concrete columns heat up to 80°

\[ \alpha_{\text{aluminum}} = 12.8 \times 10^{-6} \]
\[ \alpha_{\text{concrete}} = 6.0 \times 10^{-6} \]
\[ E_{\text{aluminum}} = 10,000 \text{ ksi} \]

Find: Compressive stress in the aluminum curtain wall panel.

Solution

Determine the free expansion of the concrete columns due to the change in temperature.

\[ \delta_{\text{concrete}} = \alpha L \Delta T = (6.0 \times 10^{-6}) (12.0) (12") \ (80° - 65°) = 0.01296 \text{ in} \]

Determine the stress in the aluminum wall panels that is required so that the expansion in the aluminum equals the expansion in the concrete.

\[ \delta_{\text{aluminum}} = \delta_{\text{concrete}} = 0.01296 \text{ in} \]
\[ \delta_{\text{aluminum}} = (\delta_{\text{aluminum}})_{\text{thermal}} - (\delta_{\text{aluminum}})_{\text{stress}} = 0.01296 \text{ in} \]
\[ = \alpha L \Delta T - P L / A E \]
\[ = \alpha L \Delta T - f L / E \]

\[ \delta_{\text{aluminum}} = 0.01296 = (12.8 \times 10^{-6}) (12.0) (12") (120° - 65°) - f (12) (12")/10,000 \]
\[ 0.01296 = 0.101376 - 144 f/10,000 \]
\[ 0.144 f = 0.101376 - 0.01296 = 0.088416 \]
\[ f = 6.14 \text{ ksi} (6,140 \text{ psi}) \]

The stress in the aluminum wall panel is 6.14 ksi.
5.5 Statically Indeterminate Members (Axially Loaded)

So far, it has always been possible to find the internal forces in any member of a structure by means of the equations of equilibrium (i.e. the structures were statically determinate).

• If, in any structure, the number of unknown forces exceeds the number of independent equations of equilibrium that are applicable, the structure is said to be "statically indeterminate."

• If a structure is statically indeterminate, it is necessary to write additional equations involving the geometry of the deformations (a.k.a. boundary conditions) in the members of the structure.
Example Problem - Statically Indeterminate Members (Axially Loaded)

Problem 5.21 (p. 297)

Given: Two plates (1/4” thick x 8” high) are placed on either side of a block of oak (4” wide x 8” high)
A load $P = 50,000$ lb is applied to the center of a rigid top plate.
$E_{\text{steel}} = 30 \times 10^6$ psi
$E_{\text{oak}} = 2 \times 10^6$ psi

Find: a) Stress developed in the steel and oak.
    b) Deformation resulting from the applied load $P$.

Solution

Deformation: $\delta_{\text{steel}} = \delta_{\text{oak}}$ (Recall: $\delta = P L / AE$)

Determine the forces acting in the steel and oak, knowing that the deformations in the steel and oak are equal and that the total load carried by the oak and steel must be $P = 50,000$ lb.

$\delta_{\text{steel}} = P_{\text{steel}} (8) / [2(1/4)8(30 \times 10^6)] = P_{\text{steel}} (6.667)(10^{-8})$

$\delta_{\text{oak}} = P_{\text{oak}} (8) / [4(8)(2 \times 10^6)] = P_{\text{oak}} (12.5)(10^{-8})$

$\delta_{\text{steel}} = \delta_{\text{oak}}$: $P_{\text{steel}} (6.667)(10^{-8}) = P_{\text{oak}} (12.5)(10^{-8})$

$P_{\text{steel}} = 1.875 P_{\text{oak}}$

$\sum F_y$: $P = 50,000 = P_{\text{steel}} + P_{\text{oak}}$

$50,000 = 1.875 P_{\text{oak}} + P_{\text{oak}} = 2.875 P_{\text{oak}}$

$P_{\text{oak}} = 50,000 / 2.875 = 17,391.3$ lb

$P_{\text{steel}} = 1.875 (17,391.3) = 32,608.7$ lb

a) Compute the stresses developed in the steel and in the oak.

$f_{\text{steel}} = P_{\text{steel}} / A_{\text{steel}} = 32,608.7 / [2(1/4)8] = 8,152$ psi

$f_{\text{oak}} = P_{\text{oak}} / A_{\text{oak}} = 17,391.3 / [(4)8] = 543$ psi

b) Compute the deformation resulting from the applied load $P$.

$\delta_{\text{steel}} = P_{\text{steel}} (8) / [2(1/4)8(30 \times 10^6)] = 32,608.7 (6.667)(10^{-8}) = 0.00217$ in

Check: $\delta_{\text{oak}} = P_{\text{oak}} (12.5)(10^{-8}) = 17,391.3 (12.5)(10^{-8}) = 0.00217$ in  OK